LOAD & STRESS ANALYSIS

MET 4501 — LECTURE NOTES — PROF. LEAH GINSBERG

DEFINITIONS

SYSTEM

THE WORD SYSTEM WILL BE USED TO DENOTE ANY ISOLATED PART OR PORTION OF A MACHINE OR STRUCTURE THAT WE WISH TO STUDY.

EQUILIBRIUM

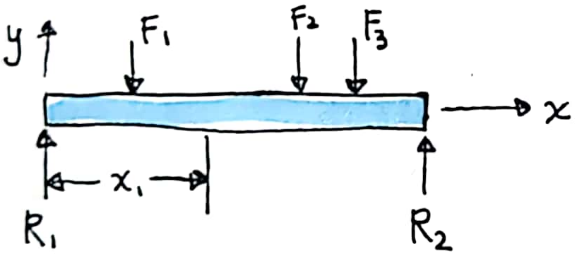
A SYSTEM IS IN EQUILIBRIUM IF IT IS MOTIONLESS OR, AT MOST, HAS CONSTANT VELOCITY SUCH THAT THERE IS ZERO ACCELERATION. THEN, THE SUM OF FORCES AND MOMENTS ACTING ON THE SYSTEM BALANCE SUCH THAT $\boxed{\Sigma F = 0 \ \& \ \Sigma M = 0}$.

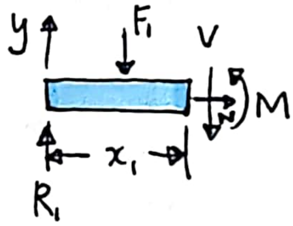
FREE-BODY DIAGRAM (FBD)

A FBD IS A GRAPHICAL REPRESENTATION USED TO VISUALIZE THE FORCES & MOMENTS ACTING ON A SYSTEM.

SHEAR FORCE & BENDING MOMENT IN BEAMS

LETS CONSIDER THE BEAM BELOW, & TAKE A CUT AT x_1:



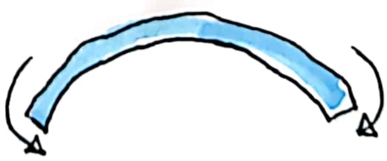


(N=0 AS THERE ARE NO AXIAL LOADS APPLIED)

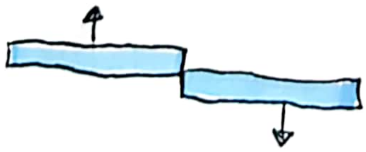
THE INTERNAL SHEAR FORCE (V) AND BENDING MOMENT (M) MUST ACT ON THE CUT SURFACE TO ENSURE EQUILIBRIUM.

SIGN CONVENTIONS FOR BENDING & SHEAR:

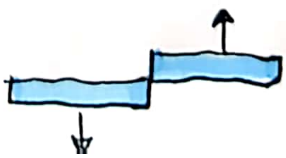
POSITIVE BENDING



NEGATIVE BENDING

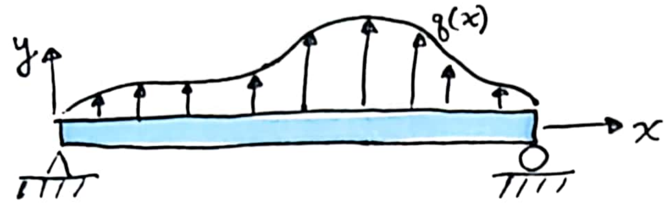


POSITIVE SHEAR



NEGATIVE SHEAR

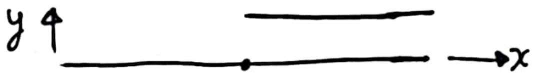
NOW, LETS CONSIDER A BEAM WITH AN ARBITRARY DISTRIBUTED LOAD. NOTE THAT $q(x)$ IS CALLED THE LOAD INTENSITY.



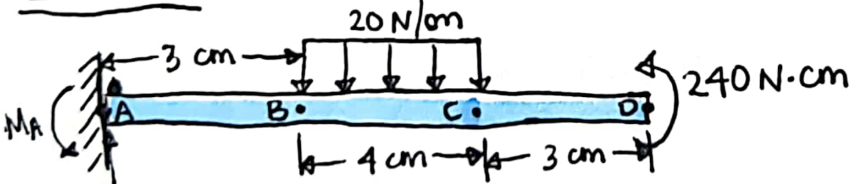
LOAD INTENSITY $q = \frac{dV}{dx} = \frac{d^2M}{dx^2}$

SHEAR FORCE $V = \frac{dM}{dx}$ $\quad V_B - V_A = \int_{x_A}^{x_B} q\,dx$

BENDING MOMENT $\quad M_B - M_A = \int_{x_A}^{x_B} V\,dx$

FUNCTION	GRAPHICAL REPRESENTATION	ORDER
CONCENTRATED MOMENT		O(-2)
CONCENTRATED FORCE		O(-1)
UNIT STEP		O(0)
RAMP		O(1)
PARABOLIC		O(2)

EXAMPLE : DRAW SHEAR & MOMENT DIAGRAMS FOR THE CANTILEVER BEAM



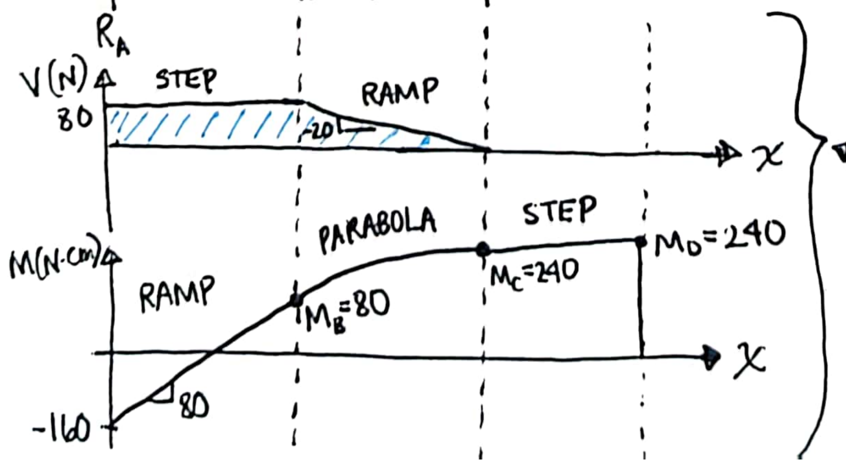
FIRST, SOLVE FOR REACTIONS AT A.

$\Sigma F = 0$: $R_A - (20\frac{N}{cm})(4\text{ cm}) = 0$

$\boxed{R_A = 80\text{ N}}$

$\Sigma M_A = 0$: $M_A - (20\frac{N}{cm})(4\text{ cm})(5\text{ cm}) + 240\text{ N}\cdot\text{cm} = 0$

$\boxed{M_A = -160\text{ N}\cdot\text{cm}}$



THEN, PLOT V & M CURVES

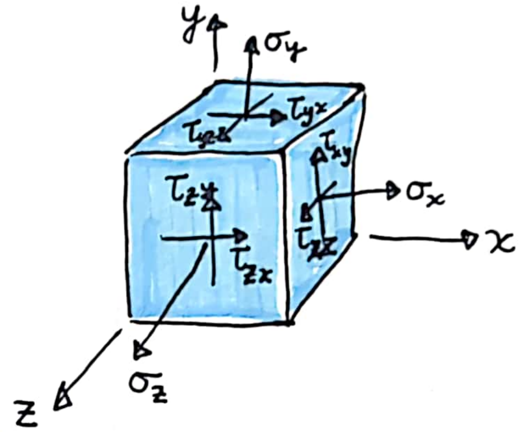
$\boxed{\begin{array}{l} V_{max} = 80\text{ N} \\ M_{max} = 240\text{ N}\cdot\text{cm} \end{array}}$

STRESS COMPONENTS

A COMPLETE STATE OF STRESS IS DEFINED BY NINE STRESS COMPONENTS, σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, and τ_{zy}.

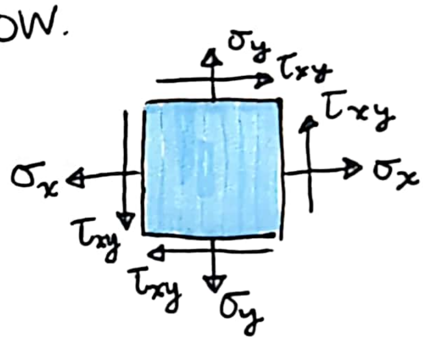
IN MOST CASES (WHEN ANGULAR MOMENTUM IS CONSERVED), "CROSS-SHEARS" ARE EQUAL.

$$\tau_{yx} = \tau_{xy}$$
$$\tau_{zy} = \tau_{yz}$$
$$\tau_{xz} = \tau_{zx}$$

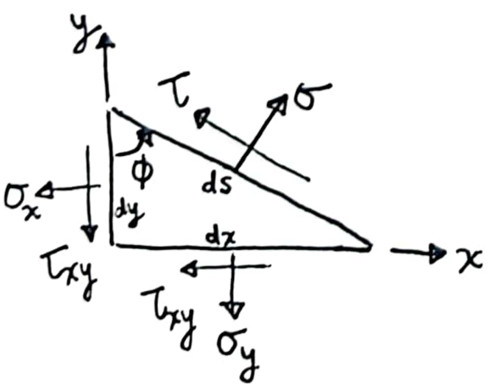


A STATE OF $\boxed{\text{PLANE STRESS}}$ OCCURS WHEN STRESSES ON ONE SURFACE ARE ZERO, AS SHOWN BELOW.

$$\sigma_z = 0$$
$$\tau_{xz} = \tau_{zx} = 0$$
$$\tau_{yz} = \tau_{zy} = 0$$



NOW, LETS TAKE A CUT THROUGH THE ELEMENT ABOVE AT AN ANGLE (ϕ) AND APPLY EQUILIBRIUM (SIMILAR TO HOW WE DID WHEN CUTTING THROUGH A BEAM).



$$\Sigma F_x = 0: \ \sigma_x dy + \tau_{xy} dx + \ldots$$
$$\Sigma F_y = 0: -\sigma_y dx - \tau_{xy} dy + \ldots$$
$$\Downarrow$$

$$\boxed{\begin{aligned} \sigma &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\phi + \tau_{xy}\sin 2\phi \\ \tau &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\phi + \tau_{xy}\cos 2\phi \end{aligned}}$$

THESE ARE CALLED THE PLANE-STRESS TRANSFORMATION EQNS.

DIFFERENTIATING THE PLANE-STRESS TRANSFORMATION EQNS WITH RESPECT TO ϕ AND SETTING THE RESULT EQUAL TO 0 MAXIMIZES σ & τ AND GIVES:

$\binom{\text{MAXIMIZING}}{\sigma}$ $\tan 2\phi_p = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$ $\xrightarrow{\text{WHICH CAN BE REWRITTEN AS:}}$ $\dfrac{\sigma_x - \sigma_y}{2}\sin 2\phi_p - \tau_{xy}\cos 2\phi_p = 0$

$\Rightarrow \tau = 0$

$\binom{\text{MAXIMIZING}}{\tau}$ $\tan 2\phi_s = -\dfrac{\sigma_x - \sigma_y}{2\tau_{xy}}$ $\xrightarrow{\text{WHICH CAN BE REWRITTEN AS:}}$ $\dfrac{\sigma_x - \sigma_y}{2}\cos 2\phi_s + \tau_{xy}\sin 2\phi_s = 0$

$\Rightarrow \sigma = \dfrac{\sigma_x + \sigma_y}{2}$

THIS MEANS THAT WHEN σ IS MAXIMIZED, $\tau = 0$, AND WHEN τ IS MAXIMIZED, σ IS THE AVERAGE OF σ_x AND σ_y.

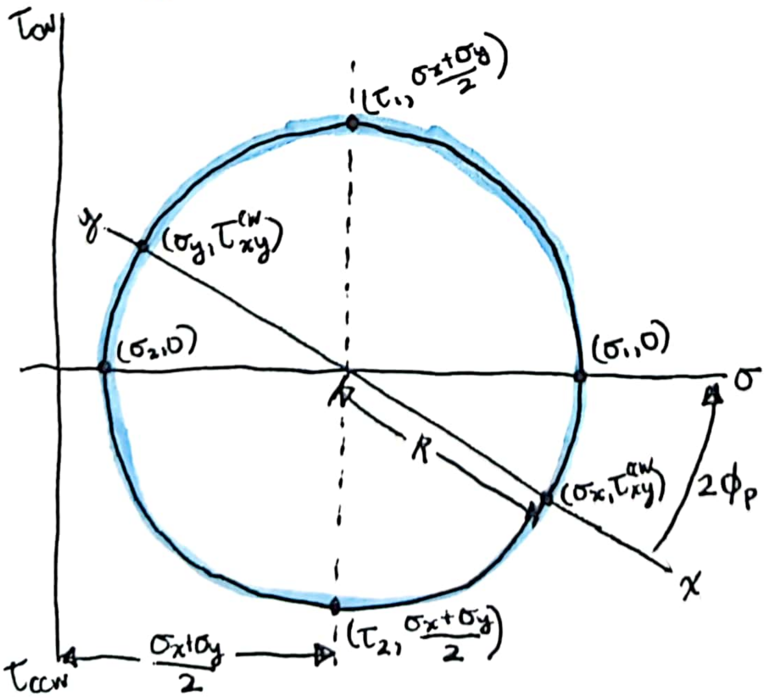
SUBSTITUTING ANGLES $2\phi_p$ AND $2\phi_s$ INTO THE PLANE STRESS TRANSFORM. EQNS YIELDS OUR PRINCIPAL STRESSES σ_1 AND σ_2.

$$\boxed{\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

AND THE EXTREME-VALUE SHEAR STRESSES, τ_1 AND τ_2.

$$\boxed{\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

MOHR'S CIRCLE



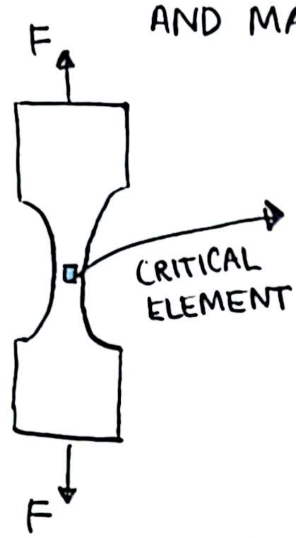
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

IN THE KITCHEN,
- CLOCK IS ABOVE
- COUNTER IS BELOW

(TO HELP REMEMBER τ^{cw} IS POSITIVE AND τ^{ccw} IS NEGATIVE)

EXAMPLE: CONSIDER A DOGBONE TENSILE SAMPLE SUBJECTED TO AN AXIAL LOAD $F = 10{,}000\,N$. THE SAMPLE IS 10 mm WIDE AND 5 mm THICK AT ITS SMALLEST CROSS-SECTION.

USE MOHR'S CIRCLE TO DETERMINE THE PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS.

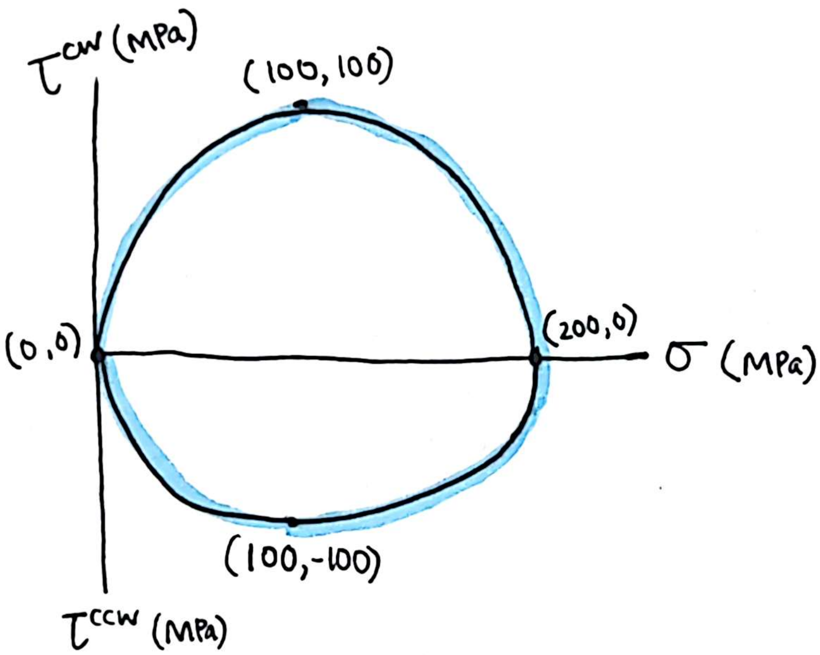


$$\sigma_y = \frac{F}{A} = \frac{10{,}000\,N}{(10\,mm)(5\,mm)}$$

$$\sigma_y = 200\text{ MPa}$$

$$\sigma_x = 0$$

$$\tau_{xy} = 0$$



$$\sigma_1 = 200\text{ MPa}$$

$$\sigma_2 = 0$$

$$\tau_{max} = 100\text{ MPa}$$

WHY THIS MATTERS: DUCTILE MATERIALS FAIL IN SHEAR (TYPICALLY), WHILE BRITTLE MATERIALS TYPICALLY FAIL DUE TO MAXIMUM PRINCIPAL STRESSES.

NOW, LET'S SAY OUR LOSS-OF-FUNCTION STRESS IS 200 MPa. THEN, IF OUR DOGBONE IS MADE OUT OF A DUCTILE MATERIAL, OUR SAFETY FACTOR IS:

$$n = \frac{200\text{ MPa}}{100\text{ MPa}} = 2.0$$

WHILE IF IT IS MADE OUT OF A BRITTLE MATERIAL,

$$n = \frac{200\text{ MPa}}{200\text{ MPa}} = 1.0 \text{ (FAILURE PREDICTED)}$$